we define inductively

<u>integral shuffle</u>

$x, y = a$ or b, $w, w' \in \mathrm{Word}(a,b)$

$$x w ш y w' := x(w ш y w') + y(x w ш w')$$

$$w ш 1 = 1 ш w = w$$

<u>series shuffle</u>

$$z_k = a^{k-1} b$$

$$z_k w * z_l w' := z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w')$$

$$z_k * 1 = 1 * z_k = z_k$$

$\forall w, w' \in a\,\mathrm{Word}(a,b)\,b$

$$w ш w' \big|_{T=1} = w * w' \big|_{T=1}$$

<u>double shuffle rel'n</u>

<u>regularized do</u>

<u>e.g.</u> $(1, 2)$

$\zeta(1)\zeta(2$

$b ш ab$

$= bab + 2abb$

$\zeta($

<u>e.g.</u> $\zeta(4) = a^3 b\big|_{T=1} = a b^3\big|_{T=1}$

$= \zeta(1,1,2)$

<u>§1.3 double shuffle rel'n</u>

<u>e.g.</u>

$$\zeta(2)\zeta(2) = \sum_{n>0} \frac{1}{n^2} \sum_{m>0} \frac{1}{m^2} = \sum + \sum + \sum = 2\zeta(2,2$$

$\Big(Z_n Z$

$Z_k * 1 = 1 * Z_k = Z_k$

$\forall w, w' \in \mathfrak{a}\,\mathrm{Word}(a,b)\,b$

$$w \sqcup\!\sqcup\, w' \big|_{T=1} = w * w' \big|_{T=1}$$

<u>double shuffle rel'n</u>

<u>regularized double shuffle rel's</u>

<u>e.g.</u> $\cancel{(1,\infty)}$

$$\zeta(1)\zeta(2) = \zeta(1,2) + \cancel{\zeta(2,1)} + \zeta(3)$$

$b \sqcup\!\sqcup\, ab$

$= bab + 2abb$

$$\| \quad \cancel{\zeta(2,1)} + 2\zeta(1,2) \qquad \cancel{(1/\infty)}$$

$$\Rightarrow \zeta(1,2) = \zeta(3)$$

we can justify this argument by studying the asymptotic behaviour of $\int_0^{1-\varepsilon}$ $(\varepsilon \to 0)$ & $\sum_{n_1 < \cdots < n_k < X}$ $(X \to \infty)$

<u>e.g.</u>

$$\zeta(2)\zeta(2) = \sum_{n>0} \frac{1}{n^2} \sum_{m>0} \frac{1}{m^2} \qquad \left(Z_n Z_m \subset Z_{n+m} \right) \qquad = \int_{0<x_1<x_2<y_1<y_2<1} + \int_{0<x_1<y_1<x_2<y_2<1}$$

<u>integral shuffle</u>

$$\| \quad = \sum_{n<m} + \sum_{m<n} + \sum_{m=n>0} = 2\zeta(2,2) + \zeta(4) \qquad + \int_{0<\cdots<1} + \int_{0<x_1<y_1<y_2<x_2<1}$$

series shuffle

$dx\, dx_2\, dy_1\, dy_2$

§2. associator relations (2-, 3-, 5-cycle relations)

a little of computations

§2.1 KZ-eq

Knizhnik - Zamolodchikov

$$\frac{dG(z)}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) G(z)$$

$G(z)$: $\mathbb{C}\langle\langle A,B\rangle\rangle$-valued (multi) analytic fct on $\mathbb{C}$

non-comm. formal power series ring

$$G(z) = \sum_{W:\text{ word of } A,B} G_W(z)\, W$$

analytic on $\mathbb{C}$

coeff. of 1 : $\frac{dG_1(z)}{dz} = 0$

coeff. of B : $\frac{dG_B(z)}{dz} = \frac{1}{z-1} G_1(z) =$

coeff. of AB : $\frac{dG_{AB}(z)}{dz} = \frac{1}{z} G_B(z) =$

coeff. of A^2B : $\frac{dG_{A^2B}(z)}{dz} = \frac{1}{z} G_{AB}(z)$

coeff. of BA^2B : $\frac{dG_{BA^2B}(z)}{dz} = \frac{1}{z-1} G$

↑ KZ eq has only regular singularities along $0, 1, \infty$

general theory → $\exists!$ solution if an asymptotic behavior is prescribed at $0, 1, \infty$

$\exists!$ $G_0(z)$: sol of KZ-eq.

$G_0(z) \approx z^A :=$

..., 5-cycle relations)

$G_\circ(z) = \sum_{\substack{W: \text{word} \\ \text{of } A, B}} G_W(z)\, W$ ← analytic on $\mathbb{C}$

analytic fct on $\mathbb{C}$

$\frac{dz}{z}, \frac{dz}{z-1}$: basis of $H^1(\mathbb{P}^1, \Omega^1(\log D))$

$A \longmapsto (0)$

$B \longmapsto (1)$

cartan-Mauer form

$(\Omega^1)^{\otimes}\Omega^1 = \mathrm{End}(\Omega^1)$

"non-commutative" Λ ... id

a bit of computations

coeff. of 1 : $\frac{dG_1(z)}{dz} = 0 \rightsquigarrow G_1(z) = \text{const} \underset{\text{put}}{=} 1$

coeff. of B : $\frac{dG_B(z)}{dz} = \frac{1}{z-1} G_1(z) = \frac{1}{z-1} \rightsquigarrow G_B(z) = \underbrace{\log(1-z)}_{= -Li_1(z)} + \underbrace{\text{const}}_{\text{put } = 0}$

coeff. of AB : $\frac{dG_{AB}(z)}{dz} = \frac{1}{z} G_B(z) = -\frac{1}{z} Li_1(z) \rightsquigarrow G_{AB}(z) = -Li_2(z) + \underbrace{\text{const}}_{= 0 \text{ put}}$

coeff. of A^2B : $\frac{dG_{A^2B}(z)}{dz} = \frac{1}{z} G_{AB}(z) = -\frac{1}{z} Li_2(z) \rightsquigarrow G_{A^2B}(z) = -Li_3(z) + \underbrace{\text{const}}_{= 0 \text{ put}}$

coeff. of BA^2B : $\frac{dG_{BA^2B}(z)}{dz} = \frac{1}{z-1} G_{A^2B}(z) = -\frac{1}{z-1} Li_3(z) \rightsquigarrow G_{BA^2B}(z) = Li_{3,1}(z) + \underbrace{\text{const}}_{\text{put } = 0}$ . . . .

$\exists !$ $G_0(z)$: sol. of KZ s.t.

$G_0(z) \approx \ldots (A \log z)^4$

$\zeta(3)$

(3)

we can justify this argument by studying the asymptotic behavior of $\int_0^{1-\varepsilon}$ $(\varepsilon \to 0)$

& $\sum_{n_1 < \cdots < n_k < X}$ $(X \to \infty)$

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KZ eq has only regular singularities along $0, 1, \infty$

general theory $\longrightarrow$ $\exists!$ solution if an asymptotic behavior is prescribed at $0, 1, \infty$

$\exists!$ $G_0(z)$: sol of KZ-eq s.t.

$G_0(z) \approx z^A = \sum_{n\geq 0} \frac{1}{n!}(A \log z)^n \quad (z \to 0)$

($G_0(z) z^{-A}$ is hol in a nbd of 0 & $=1$ at $z=0$)

$\exists!$ $G_1(z)$: : :

$G_1(z) \approx (1-z)^B \quad (z \to 1)$

$G(z) = \sum_{W \text{ word of } A, B} G_W(z) W$

coeff of

$\frac{dz}{z}, \frac{dz}{z-1}$

dz

$\exists!$ $G_0(z)$: sol of KZ-eq. s.t.

$$G_0(z) \approx z^A := \sum_{n\geq 0} \frac{1}{n!}(A \log z)^n \quad (z \to 0)$$

($G_0(z) z^{-A}$ is hol in a nbd of 0 & $=1$ at $z=0$)

$\exists!$ $G_1(z)$: " "

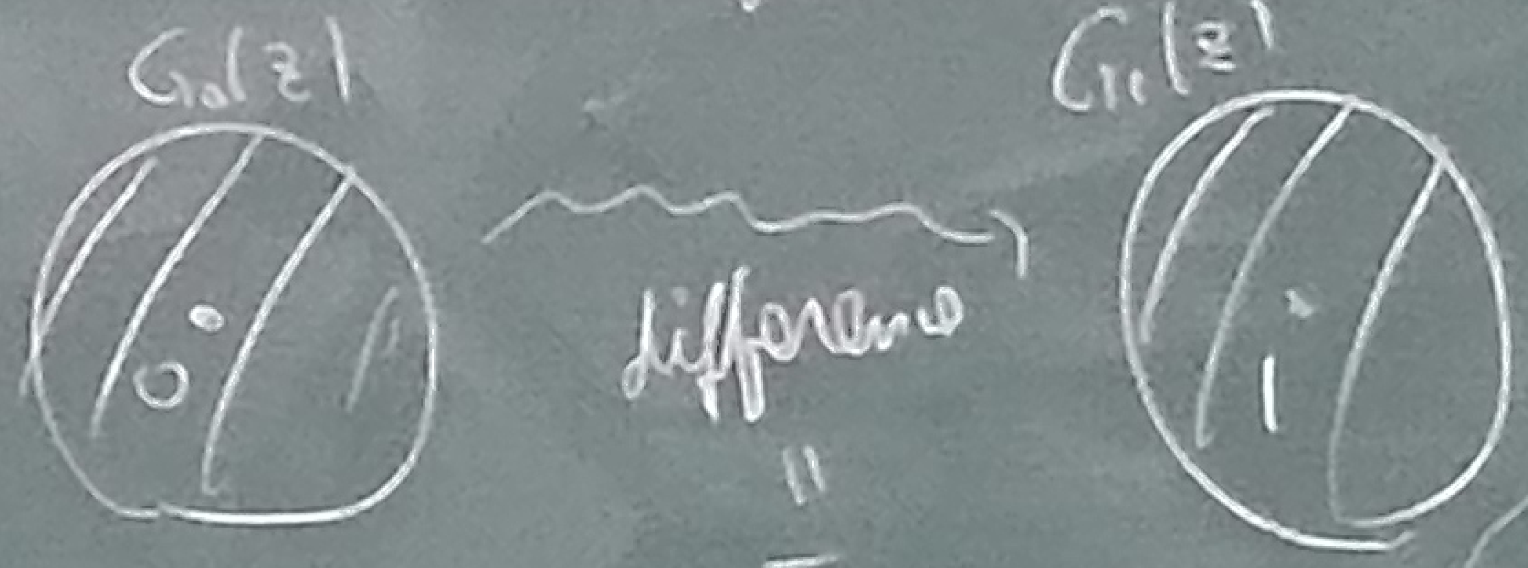
$$G_1(z) \approx (1-z)^B \quad (z \to 1)$$

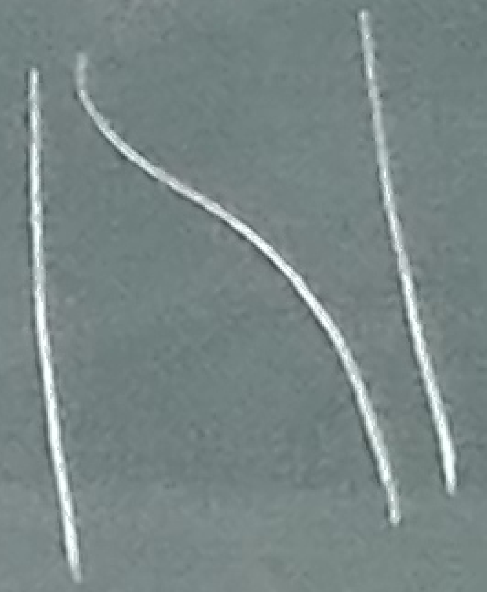
Def

$$\Phi_{KZ}(A,B) := G_1(z)^{-1} G_0(z) \in \mathbb{C}\langle\langle A,B \rangle\rangle$$

← indep. of z (both of G_0, G_1 satisfy KZ-eq.)

Drinfeld associator





$(A\otimes B)\otimes C$

$\downarrow \wr$

$A\otimes(B\otimes C)$

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^{n-1}}{n!} \quad e^t = T$$

$$\frac{T}{T-1}$$

Lem $\Phi_{KZ}(A,B) = \sum_{\substack{W:\text{word} \\ \{A,B\}}} (\Phi_{KZ})_W \, W$

$\Rightarrow (\Phi_{KZ})_{A^{k_1-1}B \cdots A^{k_d-1}B} = (-1)^d \zeta(k_1, \dots, k_d)$

§2-2 2-, 3-cycle relations

2-cycle rel'n

homotopic to 0

symmetry

$t \mapsto 1-t$

of $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$\Phi_{KZ}(A,B)\,\Phi_{KZ}(B,A) = 1$

2-cycle rel'n

later $(A \otimes B) \otimes C$

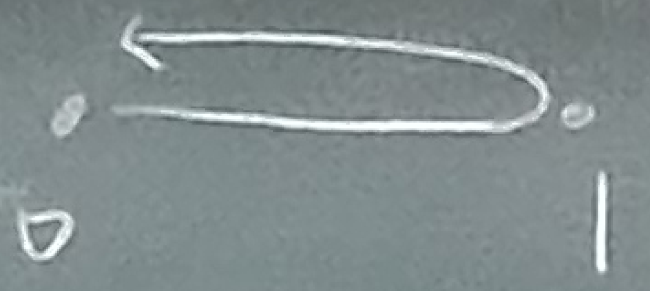
$\exists!$ $G_0(z)$: sol. of KZ-eq. s.t.

$G_0(z) \sim z^A := \sum_{n \geq 0} \frac{1}{n!}(A \log z)^n \quad (z \to 0)$

§2-2 2-, 3-cycle relations

2-cycle rel'n

homotopic to 0



asymmetry

$T \longmapsto 1-T$ of $\mathbb{P}^1 \setminus \{0,1,\infty\}$

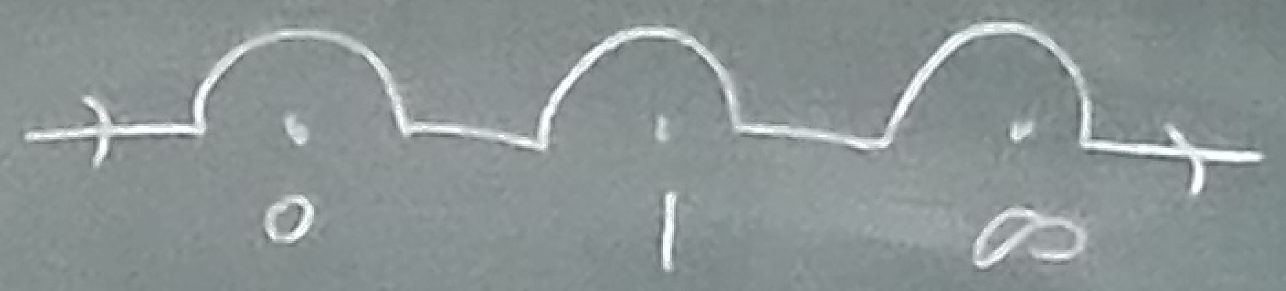
$$\Phi_{KZ}(A,B)\,\Phi_{KZ}(B,A)=1$$

2-cycle rel'n

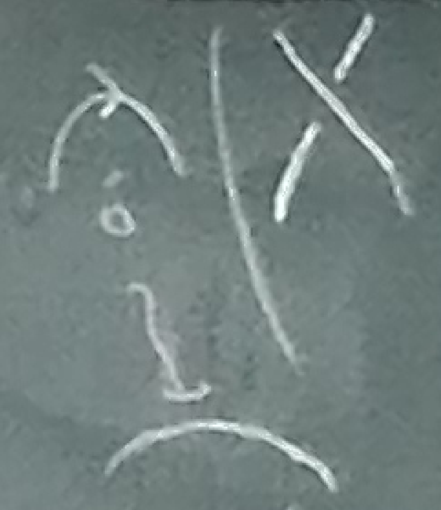
3-cycle rel'n

(integral lattice model)

homotopic to 0



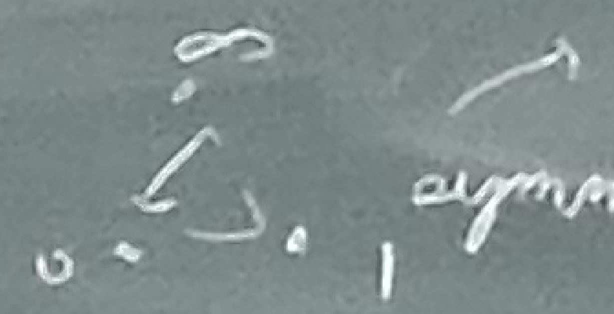
R-matrix



$$C := (AB)^{-1}$$

$$\Rightarrow \Phi_{KZ}(C,A)\,e^{\pi i C}\,\Phi_{KZ}(B,C)\,e^{\pi i B}\,\Phi_{KZ}(A,B)\,e^{\pi i A}=1$$

3-cycle relation



asymmetry

$A\otimes B$
$\downarrow$
$B\otimes A$

§3 5-cycle relation

$M_{0,4}$: moduli of curves of genus $=0$ w/ (ordered) 4 marked pts

$\cong \{(\mathbb{P}^1, (a,b,c,d))\}$ $\quad (a,b,c,d) \longmapsto (0,1,\infty,x)$, $x \neq 0,1,\infty$

$M_{0,5}$: : w/ (ordered) 5 marked pts

$(a,b,c,d,e) \longmapsto (0,1,\infty, \ldots$

<u>later</u> cf.

$$\left(\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\Phi_{A,B,C}", "\sim"'] \arrow[d, "R_{A,B}\otimes id_C"', "\wr"] & A\otimes(B\otimes C) \arrow[d, "\wr"', "id_A\otimes R_{B,C}"] \\
(B\otimes A)\otimes C \arrow[d, "\Phi_{B,A,C}"', "\wr"] \arrow[r, phantom, "\circlearrowright"] & A\otimes(C\otimes B) \\
B\otimes(A\otimes C) & (A\otimes C)\otimes B \arrow[l, "\sim", "R_{A\otimes C,B}"'] \arrow[u, "\wr", "\Phi_{A,C,B}"']
\end{tikzcd}\right)$$

<u>§2-3</u> 5-cycle relation

$M_{0,4}$: moduli of curves of genus $= 0$
w/ (ordered) 4 marked pts

$\| \wr \; (\mathbb{P}^1, (a,b,c,d))$ $\quad (a,b,c,d) \longmapsto (0,1,\infty,x)$, $\; x \neq 0,1,\infty$

$\mathbb{P}^1 \setminus \{0,1,\infty\} \ni x$